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Completeness properties of the generalized compact-open topology on partial functions with closed domains

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Abstract

The primary goal of the paper is to investigate the Baire property and weak α-favorability for the generalized compact-open topology τ_C on the space $\mathcal{P}$ of continuous partial functions $f: A \to Y$ with a closed domain $A \subset X$. Various sufficient and necessary conditions are given. It is shown, e.g., that $(\mathcal{P}, \tau_C)$ is weakly α-favorable (and hence a Baire space), if X is a locally compact paracompact space and Y is a regular space having a completely metrizable dense subspace. As corollaries we get sufficient conditions for Baireness and weak α-favorability of the graph topology of Brandi and Ceppitelli introduced for applications in differential equations, as well as of the Fell hyperspace topology. The relationship between τ_C, the compact-open and Fell topologies, respectively is studied; moreover, a topological game is introduced and studied in order to facilitate the exposition of the above results. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

Perhaps the first to consider a topological structure on the space of partial maps was Zaremba in 1936 [27] and then Kuratowski in 1955 [22], who studied the Hausdorff metric topology on the space of partial maps with compact domain. Ever since these early papers, spaces of partial maps have been studied for various purposes; in particular, the importance

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of studying topologies on partial maps has been pointed out by Filippov in his paper [12]. This observation complements the recent upsurge of various useful applications of partial maps in differential equations (see, e.g., [5,12,13,26]), in mathematical economics [3], in convergence of dynamic programming models [23] and other fields [1,2,4]; the paper of Künzi and Shapiro [20] on simultaneous extensions of partial maps with compact domains should also be mentioned here.

The so-called *generalized compact-open topology* τ_C on the space of continuous partial maps with closed domains has been especially recognized in this context (cf. [5,3,23]), whence the interest in establishing properties of this topology. Separation axioms for τ_C were characterized in [18], further, (complete) metrizability and second countability of τ_C were investigated in [19]. It is the purpose of this paper to investigate other completeness-type properties, such as weak α-favorability and Baireness of τ_C, respectively (see Section1 for the definitions) and as a consequence, of a new graph topology of Brandi and Ceppitelli (Section 5). Our results (in Section 4) naturally extend those of [19] on complete metrizability of τ_C and nicely complement similar results on the compact-open topology τ_{CO} [25,24,15] and the Fell topology τ_F [28,29], respectively.

In the pursuit of our goal we explored two approaches: the first relied on getting game-theoretical conditions on X and Y that would ensure Baireness, respectively weak α-favorability of the generalized compact-open topology and then identify some natural topological structures that satisfy these conditions. The relevant topological games are introduced and studied in Section 1.

The second approach made use of some favorable properties of the restriction mapping relating τ_C to τ_F and τ_{CO}, as well as of the already known results on Baireness and weak α-favorability of τ_{CO} and τ_F. Surprisingly, the theorems resulting from these approaches, although overlap, do not follow from each other and hence could be of independent interest (see Remark 4.5). We also give necessary conditions for the generalized compact-open topology to be Baire (of second category, in fact).

Throughout the paper X and Y will be Hausdorff topological spaces, $CL(X)$ will stand for the family of nonempty closed subsets of X (the so-called hyperspace of X) and $\mathcal{K}(X)$ for the family of (possibly empty) compact subsets of X. For any $B \in CL(X)$ and a topological space Y, $C(B, Y)$ will stand for the space of all continuous functions from B to Y. A partial map is a pair (B, f) such that $B \in CL(X)$ and $f \in C(B, Y)$. Denote by $\mathcal{P} = \mathcal{P}(X, Y)$ the family of all partial maps. Define the so-called *generalized compact-open topology* τ_C on $\mathcal{P}$ as the topology having subbase elements of the form

$$[U] = \{(B, f) \in \mathcal{P}\colon\ B \cap U \neq \emptyset\},$$

$$[K : I] = \{(B, f) \in \mathcal{P}\colon\ f(K \cap B) \subset I\},$$

where U is open in X, $K \in \mathcal{K}(X)$ and I is an open (possibly empty) subset of Y. We can assume that the I's are members of some fixed open base for Y.

A justification for calling τ_C the generalized compact-open topology can be that if (say) X is T_4 and $Y = \mathbb{R}$ (the reals), then $(\mathcal{P}, \tau_C)$ is a continuous open image (under the restriction mapping) of $(CL(X), \tau_F) \times (C(X, Y), \tau_{CO})$, where τ_{CO} is the *compact-open* *topology* [11] on $C(X,Y)$ and τ_F is the so-called *Fell topology* on $CL(X)$ having subbase elements of the form

$$V^- = \{A \in CL(X)\colon\ A \cap V \neq \emptyset\}$$

with V open in X, plus sets of the form

$$V^+ = \{A \in CL(X)\colon\ A \subset V\},$$

with V co-compact in X. It is customary [25] to use $C_k(X)$ for $(C(X,Y), \tau_{CO})$ with $Y = \mathbb{R}$ (the reals).

Both the compact-open topology and the Fell topology, respectively have been thoroughly studied and their properties are well established (cf. [25] for the compact-open topology and [8] or [21] for the Fell topology). In particular, using some previous results of McCoy and Ntantu [25], Baireness of $C_k(X)$ was characterized by Gruenhage and Ma [15] if X is a q-space; moreover, Ma showed [24] that for a locally compact X, weak α-favorability of $C_k(X)$ is equivalent to paracompactness of X.

It is also well known that the Fell hyperspace $(CL(X), \tau_F)$ is locally compact provided X is locally compact, consequently, in this case $(CL(X), \tau_F)$ is a Baire space. This result can be generalized, especially, by relaxing the requirement on Hausdorffness of X (see [28,29] for details), however, it was unknown if we can keep Hausdorffness, abandon local compactness of X and still retain Baireness of $(CL(X), \tau_F)$. We settle this problem by providing (as a byproduct of our results on τ_C) a Hausdorff non-locally compact space with a weakly α-favorable Fell hyperspace (cf. Remark 4.6).

The cardinality of the set A is denoted by $|A|$ and A^c is the complement of A. For notions not defined in the paper see [11].

1. Games

In this section we introduce several topological games played by two players α and β on a topological space (X, τ).

The first game is the well-known *Banach–Mazur game* $BM(X)$ played as follows: β starts by picking some $U_0 \in \tau \setminus \{\emptyset\}$, then α picks a $U_1 \in \tau \setminus \{\emptyset\}$ such that $U_1 \subset U_0$. In an even (respectively odd) step $n \geqslant 1$, β (respectively α) chooses a $U_n \in \tau \setminus \{\emptyset\}$ with $U_n \subset U_{n-1}$. Player α wins provided $\bigcap_{n\in\omega} U_i \neq \emptyset$, otherwise β wins (ω stands for the non-negative integers).

The second game (denoted by $BM_0(X)$) is a version of the Banach–Mazur game studied in [10]. It is played in the same manner as $BM(X)$ but the winning condition for α is that $\bigcap_{n<\omega} U_n$ is a singleton for which $\{U_n\colon\ n \in \omega\}$ is a basic system of neighborhoods (otherwise β wins).

The third game called here the *compact-open game* $KO(X)$ on (X, τ) is played as follows: β starts by picking a couple $(K_0, U_0) \in \mathcal{K}(X) \times \tau$ such that $\overline{U_0}$, the closure of U_0, is compact. Then α responds by some $V_0 \in \tau$ with compact closure that is disjoint to $K_0 \cup U_0$. In step $n \geqslant 1$, β (respectively α) chooses a couple $(K_n, U_n) \in \mathcal{K}(X) \times \tau$ (respectively a set $V_n \in \tau$) such that $\overline{U_n} \in \mathcal{K}(X)$ (respectively $\overline{V_n} \in \mathcal{K}(X)$) and

$$U_n \cap \bigcup_{i<n}(V_i \cup U_i \cup K_i) = \emptyset \quad \text{(respectively } V_n \cap \left(\bigcup_{i<n} V_i \cup \bigcup_{i\leqslant n}(U_i \cup K_i)\right) = \emptyset\text{)}.$$

Player α wins if $\{U_n\colon n \in \omega\} \cup \{V_n\colon n \in \omega\}$ is a locally finite family; otherwise, β wins.

Another game (denoted by $KO_0(X)$) is a modification of $KO(X)$, where in β's choice $K_n = \emptyset$ for all n.

Our compact-open game $KO(X)$ is closely related to the topological game $G(X)$ of Gruenhage introduced in [14], which can be described as follows: players K and L take turn in choosing compact sets; in step $n \geqslant 1$, K chooses a compact subset K_n of X and then L responds by some $L_n \in \mathcal{K}(X)$ that is disjoint to K_n. Player K wins a run of the game $G(X)$ provided $\{L_n\colon n \in \omega\}$ is a locally finite family in X; otherwise L wins.

A (*stationary*) *strategy* in these games for one of the players is a function, which picks an object for the relevant player knowing all the previous moves of the opponent as well as of his own (respectively knowing only the previous move of the opponent). A (*stationary*) *winning strategy* σ for a player is a (stationary) strategy winning for the player every run of the game compatible with σ.

The space X is called *weakly α-favorable* provided α has a winning strategy in the Banach–Mazur game $BM(X)$; further, X is *α-favorable* provided α has a stationary winning strategy in $BM(X)$. In a similar fashion, we could define weakly β-favorable and *β-favorable spaces*, respectively; however, these notions coincide (see [16]).

Proposition 1.1.
(i) *If α has a winning strategy in $KO(X)$, then so has α in $KO_0(X)$.*
(ii) *If β has a winning strategy in $KO_0(X)$, then so has β in $KO(X)$.*

Proposition 1.2. *Let $X = \bigoplus_{t\in T} X_t$ be a topological sum for some index set T such that α has a winning strategy in $KO(X_t)$ (respectively in $KO_0(X_t)$) for each $t \in T$. Then α has a winning strategy in $KO(X)$ (respectively in $KO_0(X)$).*

Proof. Let σ_t be a winning strategy for α in $KO(X_t)$ for each $t \in T$. Let n be a positive integer. Let $U_0, \ldots, U_n, V_0, \ldots, V_{n-1}$ be open sets in X with compact closure in X and $K_0, K_1, \ldots, K_n$ be compact in X. Then

$$T_0 = \left\{t \in T\colon X_t \cap \left(\bigcup_{i\leqslant n}(K_i \cup U_i) \cup \bigcup_{i<n} V_i\right) \neq \emptyset\right\}$$

is finite. Define a strategy σ for α in $KO(X)$ as follows:

$$\begin{aligned}&\sigma\big((K_0, U_0), V_0, \ldots, (K_n, U_n)\big)\\ &\quad = \bigcup_{t\in T_0} \sigma_t\big((X_t \cap K_0, X_t \cap U_0), X_t \cap V_0, \ldots, (X_t \cap K_n, X_t \cap U_n)\big)\end{aligned}$$

which is clearly a winning strategy for α in $KO(X)$. $\square$

A space is *almost locally compact* provided every nonempty open set contains a compact set with nonempty interior; X is called *hemicompact* [11], provided in the family of all compact subspaces of X ordered by inclusion there exists a countable cofinal subfamily. A space X is a *q-space* if for each $x \in X$ there is a sequence $\{G_n\}_{n\in\omega}$ of open neighborhoods of x such that whenever $x_n \in G_n$ for all n, the set $\{x_n\}_{n\in\omega}$ has a cluster point. Notice that 1st countable or locally compact (even Čech-complete) spaces are q-spaces.

Proposition 1.3.

(i) *If X is a locally compact paracompact space, then α has a winning strategy in $KO(X)$.*

(ii) *If X is an almost locally compact, non-locally compact q-space, then β has a winning strategy in $KO_0(X)$.*

Proof. (i) A locally compact, paracompact space can be written as a topological sum of σ-compact spaces (cf. the proof of Theorem 5.1.27 in [11]) and hence as a topological sum of locally compact, hemicompact spaces (see [11, Exercise 3.8.C(b)]). Then by Proposition 1.2, it suffices to prove that if X is a T_2, locally compact and hemicompact space, then α has a winning strategy in $KO(X)$.

To show this, let $U_0, \dots, U_n, V_0, \dots, V_{n-1} \in \tau$ have compact closures and $K_0, \dots, K_n \in \mathcal{K}(X)$ for some $n \in \omega$. Assume that $\mathcal{M} = \{M_i\colon i \in \omega\}$ is an increasing collection of compact sets obtained from local compactness and hemicompactness of X such that

$$\forall K \in \mathcal{K}(X)\ \exists M_i \in \mathcal{M} \text{ with } K \subset \operatorname{int} M_i.$$

Then $\bigcup_{i\leqslant n}(K_i \cup \overline{U_i}) \cup \bigcup_{i<n} \overline{V_i} \subset \operatorname{int} M_{i_n}$ for some $i_n \geqslant n$ and hence

$$V_n = (\operatorname{int} M_{i_n}) \setminus \left(\bigcup_{i\leqslant n}(K_i \cup \overline{U_i}) \cup \bigcup_{i<n} \overline{V_i} \right)$$

is an open set with compact closure.

We will show that the strategy σ defined for each $n \in \omega$ via

$$\sigma\bigl((K_0, U_0), V_0, \dots, (K_n, U_n)\bigr) = V_n$$

is a winning strategy for α in $KO(X)$.

Indeed, let $(K_0, U_0), V_0, \dots, (K_n, U_n), V_n, \dots$ be a run of $KO(X)$ compatible with σ. If $x \in X$, then $x \in \operatorname{int} M_{i_n}$ for some $i_n \geqslant n$ and $n \in \omega$. Consequently, $\operatorname{int} M_{i_n}$ is an open neighborhood of x disjoint from $\{U_i\colon i > n\} \cup \{V_j\colon j > n+1\}$, so $\{U_n\colon n \in \omega\} \cup \{V_n\colon n \in \omega\}$ is a locally finite family; thus, σ is a winning strategy for α.

(ii) Let $x \in X$ be a point with no compact neighborhood. Let $\{G_n\colon n \in \omega\}$ be a collection of countable neighborhoods of x such that whenever $x_n \in G_n$ for all n, the set $\{x_n\}_{n\in\omega}$ has a cluster point. Define a strategy σ for β in $KO_0(X)$ as follows: start by choosing a nonempty open set U_0 with compact closure contained in G_0. If $U_0, V_0, \dots, U_n, V_n$ is a run of the game $KO_0(X)$ ($n \in \omega$), then $G_{n+1} \setminus \bigcup_{i\leqslant n}(\overline{U_i} \cup \overline{V_i})$ is a nonempty open set (since $\overline{G_{n+1}}$ is not compact) and hence it contains a nonempty open set $U_{n+1} = \sigma(U_0, V_0, \dots, U_n, V_n)$

with compact closure. Pick some $x_n \in U_n$ for all n, then the sequence $\{x_n\}_{n\in\omega}$ has a cluster point y. It is clear then that every neighborhood of y intersects the collection $\{U_n\colon n\in\omega\}$ infinitely many times; thus, $\{U_n\colon n\in\omega\}\cup\{V_n\colon n\in\omega\}$ is not locally finite and σ is therefore a winning strategy for β in $KO_0(x)$. $\square$

Proposition 1.4.
(i) *If X is a locally compact space, then α has a winning strategy in $KO(X)$ iff X is paracompact.*
(ii) *If X is an almost locally compact q-space, then α has a winning strategy in $KO(X)$ iff X is paracompact and locally compact.*

Proof. In both cases, sufficiency follows from Proposition 1.3(i).

(i) *Necessity*: We will define a winning strategy θ for K in $G(X)$ given a winning strategy σ for α in $KO(X)$. Let $K_0=\emptyset$ be K's first move and let L_0 be L's response in $G(X)$. Let U_0 be an open set with compact closure containing L_0. Put $V_0=\sigma((L_0,U_0))$, $K_1=\overline{V_0}\cup\overline{U_0}$ and define $\theta(L_0)=K_1$. Suppose the game $G(X)$ has been played up to the nth step ($n\geqslant 1$): $K_0,L_0,\dots,K_n,L_n$. Clearly $L_n\cap K_n=\emptyset$; thus, by regularity and local compactness of X, there exists an open neighborhood U_n of L_n with compact closure disjoint to K_n. Put $K_{n+1}=\overline{V_n}\cup K_n\cup\overline{U_n}$, where $V_n=\sigma((L_0,U_0),V_0,\dots,V_{n-1},(L_n,U_n))$ and define $\theta(L_0,L_1,\dots,L_n)=K_{n+1}$. Then $(L_0,U_0),V_0,\dots,(L_n,U_n),V_n,\dots$ is a run of the game $KO(X)$ compatible with σ and hence $\{U_n\colon n\in\omega\}$ is a locally finite family as well as $\{L_n\colon n\in\omega\}$. It means that K has a winning strategy in $G(X)$, which in turn is equivalent to X being paracompact by a theorem of Gruenhage (see [14]).

(ii) *Necessity*: α has a winning strategy in $KO_0(X)$ by Proposition 1.1(i), so β has no winning strategy in $KO_0(x)$ and hence X is locally compact by Proposition 1.3(ii). Finally, paracompactness of X follows from Gruenhage's theorem as in (i). $\square$

In connection with Proposition 1.3(i) (also Proposition 1.4) it is worth noticing that α may have a winning strategy in $KO(X)$ even if X is not locally compact or paracompact. To show this, observe first

Lemma 1.5. *If the countable subsets of X are closed and discrete, then α has a winning strategy in $KO(X)$.*

Proof. Notice that the only compact subsets of X are the finite ones. Consequently, a winning strategy σ for α in $KO(X)$ consists of choosing the empty set regardless of β's choice. Indeed, if $(K_0,U_0),V_0,\dots,(K_n,U_n),V_n,\dots$ is a run of $KO(X)$ compatible with σ, then $V_n=\emptyset$ for all $n\in\omega$ and $U_n\subset X$ is finite for all $n\in\omega$. Hence, $C=\bigcup_{n\in\omega}U_n$ is a countable subset of X, which is discrete; thus, $\{U_n\colon n\in\omega\}\cup\{V_n\colon n\in\omega\}$ is a locally finite family. $\square$

It easily follows now from Lemma 1.5 that

Example 1.6. There exists an almost locally compact non-normal, non-q-space X such that α has a winning strategy in $KO(X)$.

Proof. Let $X = [0, 1]$. Denote by τ the natural Euclidean topology on X and put $H = \{0, 1, 1/2, \ldots, 1/n, \ldots\}$. Then

$$\big\{\{x\}\colon\ x \notin H\big\} \cup \big\{V \setminus K\colon\ V \in \tau,\ K \text{ is a countable subset of } X\big\}$$

is a base for some topology $\mathcal{O}$ on X. Of course $(X, \mathcal{O})$ is a T_2, almost locally compact space. It is easy to verify that in $(X, \mathcal{O})$ every countable set is closed and discrete, hence it is not a q-space and by Lemma 1.5, α has a winning strategy in $KO(X)$. Finally, $(X, \mathcal{O})$ is not normal, since it is not even regular. To show this, put $L = \{1, 1/2, \ldots, 1/n, \ldots\}$. Then L is a closed set in $(X, \mathcal{O})$ and $0 \notin L$, but we cannot separate $\{0\}$ and L by disjoint open sets in $(X, \mathcal{O})$. □

Compare Proposition 1.3(ii) with the following:

Example 1.7. There exists a locally compact space X such that β has a winning strategy in $KO_0(X)$.

Proof. A space with the desired properties is the so-called ladder space X on the infinite limit ordinals in ω_1 described in [15]: let $X = \omega_1$ and S stand for the infinite limit ordinals in ω_1. Define a topology on X as follows: points in $X \setminus S$ be isolated and for each $\lambda \in S$ let $\{\lambda_n \in X \setminus S\colon\ n \in \omega\}$ be an increasing sequence that is cofinal in λ (the "ladder" at λ); then the kth basic neighborhood of λ be $\{\lambda\} \cup \{\lambda_n\colon\ n \geqslant k\}$.

It is not hard to show that X is locally compact and that compact sets are at most countable. Moreover,

- β has a winning strategy in $KO_0(X)$: let $U_0 = \emptyset$ be β's first move and denote $\delta_0 = \sup(U_0 \cup V_0) + \omega$, where V_0 is α's first move. Let $f_0 : \omega \to \delta_0 \setminus S$ be a bijection, $t_{0,0} = \min\{t \in \omega\colon\ f_0(t) \notin U_0 \cup V_0\}$ and put $U_1 = \{f_0(t_{0,0})\}$. If $U_0, V_0, \ldots, U_n, V_n$ are the first $2n$ moves of the game $KO_0(X)$ $(n > 0)$, define $\delta_n = \sup(\delta_{n-1} \cup V_n) + \omega$. Let $f_n : \omega \to \delta_n \setminus (\delta_{n-1} \cup S)$ be a bijection and for each $k \in I_n = \{k \leqslant n\colon\ \operatorname{ran} f_k \setminus \bigcup_{j \leqslant n}(U_j \cup V_j) \neq \emptyset\}$ put $t_{n,k} = \min\{t \in \omega\colon\ f_k(t) \notin \bigcup_{j \leqslant n}(U_j \cup V_j)\}$. Define $U_{n+1} = \{f_k(t_{n,k})\colon\ k \in I_n\}$.

Now, if $U_0, V_0, \ldots, U_n, V_n, \ldots$ is a run of the game $KO_0(X)$ compatible with the above strategy of β, then $\lambda \setminus S \subset \bigcup_{n\in\omega}(U_n \cup V_n)$, where $\lambda = \sup \bigcup_{n\in\omega}(U_n \cup V_n) \in S$. Consequently, all the neighborhoods of λ will meet infinitely many of U_n's or V_n's. □

Finally, we list some facts about the Banach–Mazur game $BM(X)$ and its modification $BM_0(X)$ that will be used in the sequel:

Proposition 1.8. *X is non-β-favorable iff X is a Baire space, i.e., each countable intersection of dense and open subsets of X is dense.*

In particular, if X is weakly α-favorable, then X is a Baire space.

Proof. See [17, Theorem 3.16]. □

Proposition 1.9. *Let X be a regular space. Then α has a stationary winning strategy in $BM_0(X)$ iff α has a stationary winning strategy in $BM(X)$ and X contains a residual completely metrizable subspace.*

In particular, if a regular space X contains a residual completely metrizable subspace, then α has a stationary winning strategy in $BM_0(X)$.

Proof. See [10, Theorem 2.8] for the first part. As for the second part, let X_0 be a residual (hence dense) completely metrizable subspace of a regular space X and d be a compatible complete metric for X_0. Define a stationary strategy for α in $BM(X)$ as follows: if V is nonempty open in X then $V' = X_0 \cap V$ is nonempty open in X_0 and without loss of generality assume that the d-diameter of V' is bounded. Choose a nonempty X_0-open subset U' with half the diameter of that of V' and define $\sigma(V)$ to be an X-open set such that $\sigma(V) \subset V$ and $\sigma(V) \cap X_0 \subset U'$. Then completeness of (X_0, d) implies that α wins every game of $BM(X)$ compatible with σ. □

2. π-bases for the generalized compact-open topology

A collection $\mathcal{C}$ of nonempty open sets is a *π-base* for a topological space, provided each open set contains an element from $\mathcal{C}$. A topological space X is *quasi-regular*, provided nonempty opens subsets of X contain the closure of a nonempty open subset of X.

Proposition 2.1.

(i) *The collection $\mathcal{B}$ of the sets*

$$[K_0 : \emptyset] \cap \bigcap_{i \leqslant n'} [U_i] \cap \bigcap_{n' < i \leqslant n} \big([U_i] \cap [\overline{U_i} : I_i]\big) \tag{1}$$

with $n \geqslant 1$, $0 \leqslant n' < n$, $\emptyset \neq U_i \subset X$ open, $K_0, \overline{U}_{n'+1}, \ldots, \overline{U}_n \in \mathcal{K}(X)$, $K_0, U_0, \ldots, U_n$ pairwise disjoint and $\emptyset \neq I_i \subset Y$ open (for $n' < i \leqslant n$), forms a π-base for τ_C.

(ii) *If X is quasi-regular, a π-base $\mathcal{B}$ can be formed as in* (1) *with $\overline{U_0}, \ldots, \overline{U_n}$ pairwise disjoint in addition.*

(iii) *If X is almost locally compact, then the collection $\mathcal{B}_0$ of the sets*

$$[K_0 : \emptyset] \cap \bigcap_{i \leqslant n} \big([U_i] \cap [\overline{U_i} : I_i]\big) \tag{1$'$}$$

with $n \geqslant 1$, $K_0, \overline{U_i} \in \mathcal{K}(X)$, $\emptyset \neq U_i \subset X$ open, K_0, U_i pairwise disjoint for $i \leqslant n$ and $\emptyset \neq I_i \subset Y$ open $(i \leqslant n)$, forms a π-base for τ_C.

Proof. (i) Let

$$\boldsymbol{V} = [L_0 : J_0] \cap \bigcap_{j=1}^{m} \big([V_j] \cap [L_j : J_j]\big)$$

be a nonempty τ_C-basic set, where $J_0 = \emptyset$ and $J_j \neq \emptyset$ for all $1 \leqslant j \leqslant m$. Let

$$L_{00} = \bigcup_{A \in \mathcal{A}} \bigcap_{j \in A} L_j,$$

where $\mathcal{A} = \{A \subset \{0, 1, \ldots, m\}\colon\ A \neq \emptyset \text{ and } \bigcap_{j \in A} J_j = \emptyset\}$. Observe that $L_0 \subset L_{00}$.

If $(B, f) \in \boldsymbol{V}$, then there is a $b_j \in B \cap V_j \cap L_0^c$ for all $1 \leqslant j \leqslant m$, whence $b_j \notin L_{00}$, since otherwise $f(b_j) \in \bigcap_{j \in A} J_j = \emptyset$ for some $A \in \mathcal{A}$.

Let $\{v_0, \ldots, v_n\} = \{b_j\colon\ 1 \leqslant j \leqslant m\}$. Then by Hausdorffness of X, we can find a pairwise disjoint collection of open sets $U_0', \ldots, U_n'$ such that

$$v_i \in U_i' \subset \bigcap_{v_i \in V_j \setminus L_{00}} V_j \setminus L_{00} \quad \text{for all } i \leqslant n.$$

Fix $i \leqslant n$. By induction on $1 \leqslant j \leqslant m$ construct a decreasing sequence $G_1, \ldots, G_m$ of nonempty open subsets of U_i' such that for all $1 \leqslant j \leqslant m$

$$G_j \cap L_j \neq \emptyset \ \Rightarrow\ G_j \subset L_j. \tag{2}$$

If $U_i' \subset L_1$, put $G_1 = U_i'$, otherwise let $G_1 = U_i' \setminus L_1$. Further, assume that we have already constructed $G_1, \ldots, G_j$ having property (2) for some $1 \leqslant j < m$. If $G_j \subset L_{j+1}$, put $G_{j+1} = G_j$, otherwise let $G_{j+1} = G_j \setminus L_{j+1}$. Observe by (2) that

$$\emptyset \neq G_m \subset \bigcap_{j \in D_i} L_j, \tag{3}$$

where $D_i = \{1 \leqslant j \leqslant m\colon\ G_m \cap L_j \neq \emptyset\}$. Put $U_i = G_m$ and arrange that $\{i \leqslant n\colon\ D_i = \emptyset\} = \{0, 1, \ldots, n'\}$ for some $0 < n' \leqslant n$. Then $D_i \neq \emptyset$ for each $n' < i \leqslant n$, whence $\bigcap_{j \in D_i} J_j \neq \emptyset$, since $G_m \cap L_{00} \subset U_i' \cap L_{00} = \emptyset$. In this case choose a nonempty open $I_i \subset \bigcap_{j \in D_i} J_j$.

Define

$$K_0 = L_{00} \cup \bigcap_{i \leqslant n} \left(\left(\bigcup_{j=1}^{m} L_j \right) \Big\backslash U_i \right),$$

which is clearly a compact set disjoint from $\bigcup_{i \leqslant n} U_i$. Also, by (3), $\overline{U_i}$ is compact for each $n' < i \leqslant n$.

All we need to show is that $\emptyset \neq \boldsymbol{U} \subset \boldsymbol{V}$, where $\boldsymbol{U}$ is defined in (1). Indeed, to show that $\boldsymbol{U} \neq \emptyset$, pick some $u_i \in U_i$ for each $i \leqslant n$ and $z_i \in I_i$ for every $n' < i \leqslant n$. Let $B_0 = \{u_0, \ldots, u_n\}$ and define $f_0 : B_0 \to Y$ as

$$f_0(u_i) = \begin{cases} z_{n'+1}, & \text{if } i \leqslant n' + 1,\\ z_i, & \text{if } n' + 1 < i \leqslant n. \end{cases}$$

Then $(B_0, f_0) \in \boldsymbol{U}$.

Finally, take some $(B, f) \in \boldsymbol{U}$. Then by the construction of U_i's (and U_i''s) we see that for each V_j there is a U_i with $U_i \subset V_j$, whence $(B, f) \in \bigcap_{j=1}^{m} [V_j]$. Further, $L_0 \subset K_0$, so $(B, f) \in [L_0 : \emptyset]$. Moreover, it follows from $B \cap K_0 = \emptyset$ that $B \cap L_j \neq \emptyset$ implies $B \cap L_j \subset \bigcup_{i \leqslant n} U_i$.

Consequently, the set $C = \{i \leqslant n\colon\ B \cap L_j \cap U_i \neq \emptyset\} \subset \{n'+1, \ldots, n\}$ is nonempty. Thus, $D_i \neq \emptyset$ for all $i \in C$, which means, by (3), that $U_i \subset L_j$ for all $i \in C$. Consequently, $I_i \subset J_j$ for all $i \in C$. Now using that $(B, f) \in [\overline{U_i} : I_i]$ for all $n' < i \leqslant n$, we have

$$f(B \cap L_j) = \bigcup_{i \in C} f(B \cap L_j \cap U_i) \subset \bigcup_{i \in C} f(B \cap \overline{U_i}) \subset \bigcup_{i \in C} I_i \subset J_j,$$

so $(B, f) \in [L_j : J_j]$. Therefore, $(B, f) \in \boldsymbol{V}$.

(ii) If $\boldsymbol{U}$ is defined via (1) and $W_i \subset X$ is a nonempty open set with $\overline{W_i} \subset U_i$ for all $i \leqslant n$, then the $\overline{W_i}$'s are pairwise disjoint. Further, the set $L_0 = K_0 \cup \bigcup_{n' < i \leqslant n} (\overline{U_i} \setminus W_i)$ is compact, so

$$\emptyset \neq \boldsymbol{W} = [L_0 : \emptyset] \cap \bigcap_{i \leqslant n'} [W_i] \cap \bigcap_{n' < i \leqslant n} \bigl([W_i] \cap [\overline{W_i} : I_i]\bigr) \in \mathcal{B} \quad \text{and} \quad \boldsymbol{W} \subset \boldsymbol{U}.$$

(iii) Almost local compactness of X provides an open set with compact closure contained in U_i (see (i)) for each $i \leqslant n'$ (denote it by U_i again), further, putting $I_i = Y$ for all $i \leqslant n'$ we can see by (i) that elements of the form (1′) form a π-base for τ_C indeed. □

Proposition 2.2. *Let* $\boldsymbol{U} = [K_0 : \emptyset] \cap \bigcap_{i \leqslant n}([U_i] \cap [\overline{U_i} : I_i])$ *and* $\boldsymbol{V} = [L_0 : \emptyset] \cap \bigcap_{j \leqslant m}([V_j] \cap [\overline{V_j} : J_j])$ *be two elements from the* π*-base* $\mathcal{B}_0$*.*

(i) *If* $\emptyset \neq \boldsymbol{U} \subset \boldsymbol{V}$ *and* $U_{i_0} \subset V_{j_0}$ *for some* $i_0 \leqslant n$ *and* $j_0 \leqslant m$ *then* $I_{i_0} \subset J_{j_0}$*.*

(ii) *If* $\emptyset \neq \boldsymbol{U} \subset \boldsymbol{V}$*, then* $K_0 \supset L_0$ *and for each* $j \leqslant m$ *there exists* $i_j \leqslant n$ *such that* $U_{i_j} \subset V_j$ *and* $I_{i_j} \subset J_j$*.*

Proof. (i) If there exists some $y_{i_0} \in I_{i_0} \setminus J_{j_0}$, pick some $x_{i_0} \in U_{i_0}$. By pairwise disjointness of the U_i's, we can choose distinct $x_i \in U_i$ for $i \neq i_0$. Now pick arbitrary $y_i \in I_i$ for $i \neq i_0$ and define $B = \{x_0, \ldots, x_n\}$ and $f : B \to Y$ via $f(x_i) = y_i$. Then $(B, f) \in \boldsymbol{U}$, but $(B, f) \notin \boldsymbol{V}$, since otherwise

$$y_{i_0} = f(x_{i_0}) \in f(B \cap U_{i_0}) \subset f(B \cap \overline{V_{j_0}}) \subset J_{j_0},$$

which is a contradiction.

(ii) Assume that there exists $b \in L_0 \setminus K_0$. Pick some $b_i \in U_i$ and $y_i \in I_i$ arbitrarily $(i \leqslant n)$; further, let $y = y_i$, if $b = b_i$ for some i and $y \in Y$ be arbitrary otherwise. Define the set $B_0 = \{b, b_0, \ldots, b_n\}$ and the function $f_0 : B_0 \to Y$ via

$$f_0(x) = \begin{cases} y_i, & \text{if } x = b_i \text{ for } i \leqslant n, \\ y, & \text{if } x = b. \end{cases}$$

Then $(B_0, f_0) \in \boldsymbol{U} \setminus \boldsymbol{V}$, which is a contradiction and hence $L_0 \subset K_0$. Suppose now that there is $j_0 \leqslant m$ such that for all $i \leqslant n$ there exists $u_i \in U_i \setminus V_{j_0}$. Pick arbitrary $z_i \in I_i$ for all $i \leqslant n$. Then for $B_1 = \{u_0, \ldots, u_n\}$ and $f_1 : B_1 \to Y$ defined as $f_1(u_i) = z_i$ $(i \leqslant n)$, we have $(B_1, f_1) \in \boldsymbol{U} \setminus \boldsymbol{V}$, a contradiction. The remaining follows from (i). □

3. Properties of the restriction mapping

The restriction mapping

$$\eta : \bigl(CL(X), \tau_F\bigr) \times \bigl(C(X, Y), \tau_{CO}\bigr) \to (\mathcal{P}, \tau_C)$$

is defined as $\eta((B, f)) = (B, f\restriction_B)$. Clearly, η is onto provided continuous partial functions with closed domain are continuously extendable over X. The following proposition gives some sufficient conditions for this:

Proposition 3.1. *There exists a base $\mathcal{V}$ for Y such that for each $A \in CL(X)$, $V \in \mathcal{V}$, every function $f \in C(A, V)$ is extendable to some $f^* \in C(X, V)$, if either of the following holds*:
(i) *X is T_4 and $Y \subset \mathbb{R}$ is an interval*;
(ii) *X is paracompact and Y is a locally convex completely metrizable space.*

Proof. (i) This is the Tietze Extension Theorem with the open intervals in Y as $\mathcal{V}$.

(ii) This is a consequence of Michael's Selection Theorem as presented in [8, Proposition 6.6.4]. Indeed, the proof goes through under our conditions as well with $\mathcal{V}$ being the convex open subsets of Y. $\square$

Proposition 3.2. *If X is a regular space, then η is continuous.*

Proof. See [18, Proposition 1.5]. $\square$

Proposition 3.3. *Let X, Y be such that partial continuous functions with closed domains are continuously extendable over X; moreover, suppose that there exists an open base $\mathcal{V}$ for Y closed under finite intersections such that for each nonempty $K \in \mathcal{K}(X)$ and $V \in \mathcal{V}$, every function $f \in C(K, V)$ is extendable to some $f^* \in C(X, V)$. Then η is an open mapping.*

Proof. Let $\boldsymbol{V} = \boldsymbol{V}_F \times \boldsymbol{V}_{CO}$ be a nonempty $\tau_F \times \tau_{CO}$-open set, where

$$\boldsymbol{V}_F = (L_0^c)^+ \cap \bigcap_{j=1}^m V_j^- \in \tau_F \quad \text{and} \quad \boldsymbol{V}_{CO} = C(X, Y) \cap \bigcap_{j=1}^m [L_j : J_j] \in \tau_{CO}$$

with $J_j \in \mathcal{V}$ for each j; further, denote $\boldsymbol{U} = [L_0 : \emptyset] \cap \bigcap_{j=1}^m ([V_j] \cap [L_j : J_j]) \in \tau_C$. Then $\eta(\boldsymbol{V}) = \boldsymbol{U}$.

Indeed, $\eta(\boldsymbol{V}) \subset \boldsymbol{U}$ is clear and we will prove that $\boldsymbol{U} \subset \eta(\boldsymbol{V})$: without loss of generality assume that each L_j intersects with $L_{j'}$ for some $j' \neq j$. For $M \subset \{1, \ldots, m\}$ put

$$L_M = \bigcap_{j \in M} L_j, \qquad J_M = \bigcap_{j \in M} J_j$$

and let $\mathcal{M} = \{M \subset \{1, \ldots, m\}\colon\ L_M \neq \emptyset$ and $L_M \cap L_j = \emptyset$ for each $j \notin M\}$. Then $J_M \in \mathcal{V}$ is nonempty for every $M \in \mathcal{M}$ (otherwise $f(x) \in J_M = \emptyset$ for each $f \in \boldsymbol{V}_{CO}$ and $x \in L_M$—a contradiction). Denote $t_0 = \max\{|M|\colon\ M \in \mathcal{M}\}$ (which is at least 2) and put $\mathcal{M}_0 = \{M \in \mathcal{M}\colon\ |M| = t_0\}$; moreover, for each $0 < t < t_0$ define

$$\mathcal{M}_t = \bigl\{M \setminus \{j\}\colon\ M \in \mathcal{M}_{t-1},\ j \in M\bigr\} \cup \bigl\{M \in \mathcal{M}\colon\ |M| = t_0 - t\bigr\}.$$

Notice that $\mathcal{M}_{t_0-1} = \{\{j\}\colon\ 1 \leqslant j \leqslant m\}$ and $|M| = t_0 - t$ for each $M \in \mathcal{M}_t$, $0 \leqslant t < t_0$.

Choose $(D, g) \in \boldsymbol{U}$. Then $D \in \boldsymbol{V}_F$ and if we construct a function $g^* \in \boldsymbol{V}_{CO}$ such that $g^*\restriction_D = g$, then $(D, g) = \eta((D, g^*)) \in \eta(\boldsymbol{V})$ and we are done. For every $M \in \mathcal{M}$, extend

$g\restriction_{D\cap L_M}$ to some $g_M \in C(L_M, J_M)$ provided $D \cap L_M \neq \emptyset$; otherwise, define $g_M(x) = y_M$ for each $x \in L_M$, where y_M is a fixed element of J_M. Observe that this defines g_M's for each $M \in \mathcal{M}_0$. Now, by induction on t, we can construct for each $0 < t < t_0$ and $M \in \mathcal{M}_t$ a function $g_M \in C(L_M, J_M)$ so that $g_M = g$ on $D \cap L_M$ and $g_M = g_{M'}$ on $L_{M'}$ for each $M' \in \mathcal{M}_{t-1}$ with $M \subset M'$.

Indeed, assume that $g_{M'}$ has been defined for all $M' \in \mathcal{M}_{t-1}$, where $0 < t < t_0$. Let $M \in \mathcal{M}_t$. If in addition $M \in \mathcal{M}$, then g_M satisfies our conditions, since there is no $M' \in \mathcal{M}_{t-1}$ containing M. Suppose therefore that $M \in \mathcal{M}_t \setminus \mathcal{M}$. Then in view of the induction hypothesis, the function

$$g'(x) = \begin{cases} g(x), & x \in D \cap L_M, \\ g_{M'}(x), & x \in L_{M'}, M' \in \mathcal{M}_{t-1}, M \subset M' \end{cases}$$

is well-defined on $D' = D \cap L_M \cup \bigcup\{L_{M'}\colon\ M' \in \mathcal{M}_{t-1},\ M \subset M'\} \subset L_M$; moreover, $g' \in C(D', J_M)$. Hence we can extend g' to some $g_M \in C(L_M, J_M)$ and our conditions will be satisfied.

Finally, using the fact that continuous partial functions with closed domains are continuously extendable over X, we can find a $g^* \in C(X, Y)$ so that $g^* = g$ on D and $g^* = g_{\{j\}}$ for each $1 \leqslant j \leqslant m$ (note that $\mathcal{M}_{t_0-1} = \{\{j\}\colon\ 1 \leqslant j \leqslant m\}$ and $L_{\{j\}} = L_j$ for each j). $\square$

Corollary 3.4.

(i) *Let X, Y be such that partial continuous functions with closed domains are continuously extendable over X; moreover, suppose that there exists an open base $\mathcal{V}$ for Y closed under finite intersections such that for each nonempty $K \in \mathcal{K}(X)$ and $V \in \mathcal{V}$, every function $f \in C(K, V)$ is extendable to some $f^* \in C(X, V)$.*
Then η is open, continuous and onto.

(ii) *If X is paracompact and Y is locally convex completely metrizable or if X is T_4 and $Y \subset \mathbb{R}$ is an interval, then η is open, continuous and onto.*

Proof. Compare Propositions 3.1–3.3. $\square$

4. Baireness and weak α-favorability of the generalized compact-open topology

Theorem 4.1. *Let X, Y be such that partial continuous functions with closed domains are continuously extendable over X; moreover, suppose that there exists an open base $\mathcal{V}$ for Y closed under finite intersections such that for each nonempty $K \in \mathcal{K}(X)$ and $V \in \mathcal{V}$, every function $f \in C(K, V)$ is extendable to some $f^* \in C(X, V)$. Then*

(i) *$(\mathcal{P}, \tau_C)$ is a Baire space, if $(CL(X), \tau_F) \times (C(X, Y), \tau_{CO})$ is a Baire space.*

(ii) *$(\mathcal{P}, \tau_C)$ is (weakly) α-favorable, if $(CL(X), \tau_F)$ as well as $(C(X, Y), \tau_{CO})$ are (weakly) α-favorable.*

Proof. (i) Use Corollary 3.4(i) and the fact that continuous, open and onto mappings preserve Baire spaces (see [17, Theorem 4.7]).

(ii) (Weakly) α-favorable spaces are productive and are preserved by continuous, open and onto mappings, hence Corollary 3.4(i) applies. $\square$

Theorem 4.2. *Let X be an almost locally compact space and assume that α has a stationary winning strategy in $BM_0(Y)$. Then*

(i) *$(\mathcal{P}, \tau_C)$ is a Baire space if β has no winning strategy in $KO(X)$;*

(ii) *$(\mathcal{P}, \tau_C)$ is weakly α-favorable if α has a winning strategy in $KO(X)$.*

Proof. Let σ_Y be a stationary winning strategy for α in $BM_0(Y)$. Let σ_X be the function assigning to an open $\emptyset \neq U \subset X$ an open set $\emptyset \neq V \subset X$ with compact closure such that $\overline{V} \subset U$.

(i) Let σ be a strategy for β in $BM(\mathcal{P})$. We will define a strategy for β in $KO(X)$ making use of σ as follows: let

$$\boldsymbol{V}_0 = [L_{0,0} : \emptyset] \cap \bigcap_{j \leqslant m_0} \bigl([V_{0,j}] \cap [\overline{V_{0,j}} : J_{0,j}]\bigr) \in \mathcal{B}_0$$

be the first step of β in $BM(\mathcal{P})$ for some $m_0 \in \omega$. Then let (K_0, W_0) be the first step of β in $KO(X)$, where $K_0 = L_{0,0}$ and $W_0 = \bigcup_{j \leqslant m_0} V_{0,j}$. Suppose that (K_0, W_0), $W_1, (K_2, W_2), \ldots, (K_{n-1}, W_{n-1}), W_n$ are the first $n+1$ steps of the game $KO(X)$ for some *odd* $n \in \omega$. Also assume that in the game $BM(\mathcal{P})$ the first n moves were the sets $\boldsymbol{V}_0 \supset \boldsymbol{V}_1 \supset \cdots \supset \boldsymbol{V}_{n-1}$, where for each $k \leqslant n-1$

$$\boldsymbol{V}_k = [L_{k,0} : \emptyset] \cap \bigcap_{j \leqslant m_k} \bigl([V_{k,j}] \cap [\overline{V_{k,j}} : J_{k,j}]\bigr) \in \mathcal{B}_0, \tag{5}$$

with $m_0 \leqslant m_1 \leqslant \cdots \leqslant m_{n-1}$ (see Proposition 2.2(ii)). We want to make sure on each stage that β's strategy in $KO(X)$ mirrors β's strategy in $BM(\mathcal{P})$ so that for each *even* $1 \leqslant k \leqslant n-1$

$$K_k = L_{k,0} \quad \text{and} \quad W_k = \bigcup_{j \leqslant m_k} V_{k,j} \setminus \overline{\bigcup_{j \leqslant m_{k-1}} V_{k-1,j}}. \tag{6}$$

For each $j \leqslant m_{n-1}$ define

$$V_{n,j} = \sigma_X(V_{n-1,j}) \quad \text{and} \quad J_{n,j} = \sigma_Y(J_{n-1,j}) \tag{7}$$

and if $W_n \neq \emptyset$, put $V_{n,m_{n-1}+1} = \sigma_X(W_n)$ and $J_{n,m_{n-1}+1} = Y$. Finally, let

$$L_{n,0} = L_{n-1,0} \cup \bigcup_{j \leqslant m_n} (\overline{V_{n-1,j}} \setminus V_{n,j}) \in \mathcal{K}(X), \tag{8}$$

where $m_n = m_{n-1} + 1$ if $W_n \neq \emptyset$, otherwise $m_n = m_{n-1}$. Then $\boldsymbol{V}_n$ (defined as in (5) for $k = n$) is a well-defined response of α in $BM(\mathcal{P})$ (see (7), (8)). If

$$\boldsymbol{V}_{n+1} = \sigma(\boldsymbol{V}_0, \ldots, \boldsymbol{V}_n) \tag{9}$$

is the next choice of β in $BM(\mathcal{P})$ and if $\boldsymbol{V}_{n+1}$ is expressed in the form (5) for $k = n+1$ and some $m_{n+1} \geqslant m_n$, then we can define β's next step (K_{n+1}, W_{n+1}) in $KO(X)$ using (6) for $k = n+1$.

This defines a strategy for β in $KO(X)$, which is not winning by our assumption on $KO(X)$. Therefore, α can play so that the collection

$$\{W_n\colon n \in \omega\} \text{ is locally finite.}$$

We will show that β loses the corresponding game in $BM(\mathcal{P})$: for $n \in \omega$ let

$$E_{n+1} = \left\{ j \leqslant m_{n+1}\colon V_{n+1,j} \cap \left(\bigcup_{j' \leqslant m_n} V_{n,j'} \right) = \emptyset \right\}.$$

Observe by (8) that for $j \leqslant m_{n+1}$ either $V_{n+1,j} \subset \bigcup_{j' \leqslant m_n} V_{n,j'}$ or $j \in E_{n+1}$. Without loss of generality we can assume that $E_{n+1} \neq \emptyset$ for all $n \in \omega$ and that for all $j \notin E_{n+1}$ ($j \leqslant m_{n+1}$) there exists some $j' \leqslant m_n$ such that $V_{n+1,j} \subset V_{n,j'}$.

Then we can define the following collections of pairwise disjoint sets:

$$\mathcal{W}_{0,0} = \{V_{0,j}\colon j \leqslant m_0\} \quad \text{and}$$

$$\mathcal{W}_{n+1,n+1} = \{V_{n+1,j}\colon j \in E_{n+1}\} \quad \text{for } n \in \omega.$$

Notice that $W_n = \bigcup \mathcal{W}_{n,n}$ for all $n \in \omega$. For $k > n$ put

$$\mathcal{W}_{n,k} = \{V_{k,j}\colon j \leqslant m_k \text{ and } V_{k,j} \subset W_n\}.$$

Then for all $k \in \omega$

$$\bigcup_{n \leqslant k} \mathcal{W}_{n,k} = \{V_{k,j}\colon j \leqslant m_k\} \tag{10}$$

and $\mathcal{W}_{n,k+1}$ is a refinement of $\mathcal{W}_{n,k}$ for all $k \geqslant n$. In view of (7)

$$B_n = \bigcap_{k > (n-1)/2} \overline{\bigcup \mathcal{W}_{n,2k+1}} = \bigcap_{k > (n-1)/2} \left(\bigcup \mathcal{W}_{n,2k} \right) \tag{11}$$

is a nonempty closed subset of W_n for all $n \in \omega$.

Also, if $x \in B_n$, there exists a unique decreasing sequence $V_{k,j_k} \in \mathcal{W}_{n,k}$ ($k \geqslant 2n$) such that $x \in \bigcap_{k \geqslant 2n} V_{k,j_k}$. Since in view of (7), $J_{2n,j_{2n}}, \ldots, J_{k,j_k}, \ldots$ is a run of $BM_0(Y)$ compatible with σ_Y, there exists a unique $y \in \bigcap_{k \geqslant 2n} J_{k,j_k}$ for which $\{J_{k,j_k}\colon k \geqslant 2n\}$ is a basic system of neighborhoods. Let f be the function that assigns y to x in this manner; then f is defined on $B = \bigcup_{n \in \omega} B_n$.

Claim 1. $B \in CL(X)$.

Proof. Indeed, it was shown that $\{W_n\colon n \in \omega\}$ is a locally finite collection, consequently, $\{B_n\colon n \in \omega\}$ is locally finite as well, since $B_n \subset W_n$ for all $n \in \omega$; thus, $B = \bigcup_{n \in \omega} B_n$ is closed. □

Claim 2. $f \in C(B, Y)$.

Proof. Let U be nonempty open in Y and $y = f(x) \in U$. Let $J_{2n,j_{2n}}, \ldots, J_{k,j_k}, \ldots$ be a decreasing sequence of open sets intersecting in $\{y\}$ that is a neighborhood-base for y.

Then there is some $k_0 \geqslant 2n$ with $y \in J_{k_0, j_{k_0}} \subset U$. Consider the set $V = B \cap V_{k_0, j_{k_0}}$, which is open in B and contains x. Further, if $x' \in V$ then there exists a unique decreasing sequence $\{V_{k, j'_k}\colon k \geqslant 2n\}$ such that $j'_{k_0} = j_{k_0}$; so, by Proposition 2.2(i),

$$f(x') \in \bigcap_{k \geqslant 2n} J_{k, j'_k} \subset J_{k_0, j'_{k_0}} = J_{k_0, j_{k_0}} \subset U.$$

It means that $f^{-1}(U)$ is open in B and hence $f \in C(B, Y)$. $\square$

Claim 3. $(B, f) \in \bigcap_{n \in \omega} \boldsymbol{V}_n$.

Proof. Fix $n \in \omega$. Since $B_k \subset W_k$, we have that $B_k \cap L_{n,0} = \emptyset$ for all $k \geqslant n$; further, if $k < n$ then $B_k \subset \bigcup \mathcal{W}_{k,n} \subset (L_{n,0})^c$. Hence $B \cap L_{n,0} = \emptyset$.

It is also clear from (10) and (11) that $B \cap V_{n,j} \neq \emptyset$ for all $j \leqslant m_n$. Finally, $f(\overline{B \cap V_{n,j}}) \subset J_{n,j}$ $(j \leqslant m_n)$ by the definition of f. $\square$

(ii) Let σ_{KO} be a winning strategy for α in $KO(X)$. Define a strategy σ for α in $BM(\mathcal{P})$ as follows: for all $k \leqslant n$ (n even) define $\boldsymbol{V}_k$ via (5), where $\boldsymbol{V}_0 \supset \boldsymbol{V}_1 \supset \cdots \supset \boldsymbol{V}_n$. For $j \leqslant m_n$ define $V_{n+1,j}$ and $L_{n+1,0}$ as in (7) and (8), respectively replacing n by $n+1$. For each $k \in \omega$, let W_k be defined as in (i) (see (6)) and put

$$V_{n+1, m_n+1} = \sigma_{KO}\big((L_{0,0}, W_0), W_1, (L_{2,0}, W_2), \ldots, W_{n-1}, (L_{n,0}, W_n)\big)$$

and let $J_{n+1,m_n+1} = Y$. Finally, for $m_{n+1} = m_n + 1$ let $\boldsymbol{V}_{n+1}$ be given by (5) with $k = n+1$ and define σ via (9).

It is not hard to show that $\boldsymbol{V}_{n+1} \subset \boldsymbol{V}_n$ and analogously to (i) we can prove (through Claims 1–3) that σ is a winning strategy for α in $BM(\mathcal{P})$. $\square$

The following corollary extends and complements results of [28,29] concerning Baireness and α-favorability of the Fell topology:

Corollary 4.3. *Let X be an almost locally compact space. Then*
(i) *$(CL(X), \tau_F)$ is a Baire space if β has no winning strategy in $KO(X)$;*
(ii) *$(CL(X), \tau_F)$ is weakly α-favorable if α has a winning strategy in $KO(X)$.*

Proof. Observe that if $Y = \{y\}$ is a singleton, then $(\mathcal{P}, \tau_C)$ is homeomorphic to $(CL(X), \tau_F)$ and hence Theorem 4.2 applies. $\square$

A collection $\mathcal{K}$ of nonempty compact subsets of X is called a moving off collection if, for any compact set $L \subset X$, there exist some $K \in \mathcal{K}$ disjoint to L. Following [15], we say that X has the *moving off property* (*MOP*) provided every moving off collection of nonempty compact sets contains an infinite subcollection which has a discrete open expansion in X.

Corollary 4.4.
(i) *Let X be a locally compact paracompact space. Let Y be a regular space having a completely metrizable residual subspace. Then $(\mathcal{P}, \tau_C)$ is weakly α-favorable.*

(ii) *Let X be a T_4, locally compact space with the MOP and $Y = \mathbb{R}$. Then $(\mathcal{P}, \tau_C)$ is a Baire space.*

Proof. (i) Compare Theorem 4.2(ii), Proposition 1.4(i) and Proposition 1.9.

(ii) If X is locally compact then the Fell topology $(CL(X), \tau_F)$ is also locally compact [8, Corollary 5.1.4] and hence weakly α-favorable; further, it has been shown in [15] that $C_k(X)$ is a Baire space if X is a locally compact space with the MOP. It is also known (see [17, Theorem 5.1(ii)]), that the product of a weakly α-favorable space and a Baire space is a Baire space; therefore, in view of Proposition 3.1(i) and Theorem 4.1(i), $(\mathcal{P}, \tau_C)$ is a Baire space. $\square$

Remark 4.5. Observe that Theorem 4.2(i) and Theorem 4.1(i) overlap but do not follow from each other. Indeed, the space from Example 1.6 is not regular, hence Theorem 4.1(i) does not apply (if Y contains at least two distinct points). However, by Theorem 4.2(i), $(\mathcal{P}, \tau_C)$ is a Baire space if (say) Y is a regular space having a dense completely metrizable subspace.

On the other hand, if X is the space from Example 1.7, then β has a winning strategy in $KO_0(X)$ (and hence in $KO(X)$ as well); thus, Theorem 4.2(i) is useless. However, X has the MOP (see [15, Example 4.1]) and it can be shown under (MA+$\neg$ CH) that X is T_4. It follows then by Corollary 4.4(ii), that under (MA+$\neg$ CH) and with $Y = \mathbb{R}$, $(\mathcal{P}, \tau_C)$ is a Baire space.

Remark 4.6. The space X from Example 1.6 also provides an example of a T_2 non-locally compact space such that $(CL(X), \tau_F)$ is weakly α-favorable (see Corollary 4.3).

Lastly, we will explore some necessary conditions for Baireness (for being of second category even) of $(\mathcal{P}, \tau_C)$.

Lemma 4.7. *Let X be an almost locally compact space and U an open subset with non-compact closure in X. Let $\mathcal{G}$ be the family of nonempty open subsets of X with compact closure contained in U and J be a nonempty open subset of Y. Then the set*

$$H(U, J) = \bigcup_{O \in \mathcal{G}} \bigl([O] \cap [\overline{O} : J]\bigr)$$

is open and dense in $(\mathcal{P}, \tau_C)$.

Proof. $H(U, J)$ is clearly open. Further, let

$$H = [K : \emptyset] \cap \bigcap_{i \leqslant n} \bigl([U_i] \cap [\overline{U_i} : I_i]\bigr)$$

with $K, \overline{U_i} \in K(X)$, $\emptyset \neq U_i \subset X$, U_i open, K, U_i $(i \leqslant n)$ pairwise disjoint and $\emptyset \neq I_i \subset Y$ open $(i \leqslant n)$, be an element of the π-base $\mathcal{B}_0$ (see $(1')$ in Proposition 2.1). For every $i \leqslant n$ choose $x_i \in U_i$ and $y_i \in I_i$. The set $L = (K \cup \bigcup_{i \leqslant n} \overline{U_i})$ is compact, thus, $U \setminus L \neq \emptyset$. There is an $O \in \mathcal{G}$ such that $\overline{O}$ is compact, $\overline{O} \subset U \setminus L$. Choose $x \in O$ and $y \in J$. Put

$B = \{x, x_0, \ldots, x_n\}$ and define f on B as follows: $f(x) = y$ and $f(x_i) = y_i$ for each $i \leqslant n$. Then $(B, f) \in H \cap H(U, J)$. □

Proposition 4.8. *Let X be an almost locally compact space and Y contain an infinite locally finite collection of open sets (e.g., Y be a non-compact paracompact space). Let $U \subset X$ be a nonempty open set with a countably compact closure. Then $\overline{U}$ is compact if $(\mathcal{P}, \tau_C)$ is of second category (i.e., countable intersections of dense open subsets are nonempty).*

In particular, an almost locally compact, countably compact space X is compact, if $(\mathcal{P}, \tau_C)$ is of second category.

Proof. Suppose that $\overline{U}$ is not compact. Let $\{J_n \subset Y\colon n \in \omega\}$ be a locally finite collection of nonempty open sets. Then Lemma 4.7 implies that, $H_n = H(U, J_n)$ is dense and open in $(\mathcal{P}, \tau_C)$ for each $n \in \omega$. Since the generalized compact-open topology τ_C is of second category, we have that $\bigcap_{n\in\omega} H_n \neq \emptyset$, hence there exists some $(C, g) \in \bigcap_n H_n$. Consequently, for every $n \in \omega$ there is $c_n \in C \cap U$ with $g(c_n) \in J_n$. Then continuity of g implies that $\{c_n\colon n \in \omega\}$ has no cluster point, a contradiction with countable compactness of $\overline{U}$. □

In view of Proposition 1.3(ii) and Proposition 1.1(ii), X is locally compact if X is an almost locally compact q-space such that β has no winning strategy in $KO(X)$. Further, by Theorem 4.2(i), if β has no winning strategy in $KO(X)$, then $(\mathcal{P}, \tau_C)$ with (say) $Y = \mathbb{R}$ is a Baire space. It may be of interest therefore to find out under what conditions does Baireness of $(\mathcal{P}, \tau_C)$ imply local compactness of X. The following proposition gives an answer in the framework of Proposition 1.3(ii):

Proposition 4.9. *Let X be an almost locally compact q-space and Y contain an infinite, locally finite collection of open sets. If $(\mathcal{P}, \tau_C)$ is of second category, then X is locally compact.*

Proof. Suppose that we can find a point $x \in X$ with no compact neighborhoods in X. Let $\{G_n\colon n \in \omega\}$ be a sequence of open neighborhoods of x such that whenever $x_n \in G_n$, then $\{x_n\colon n \in \omega\}$ has a cluster point. Further, let $\{J_n \subset Y\colon n \in \omega\}$ be a locally finite collection of nonempty pairwise disjoint open sets.

By Lemma 4.7, the sets $H_n = H(G_n, J_n)$ are dense and open in $(\mathcal{P}, \tau_C)$ for each $n \in \omega$; thus, there exists some $(C, g) \in \bigcap_{n\in\omega} H_n$. If $x_n \in C \cap G_n$ is such that $g(x_n) \in J_n$ for all $n \in \omega$, then the net $\{x_n\colon n \in \omega\}$ has a cluster point $c \in C$, which contradicts continuity of g. □

Remark 4.10. Being a q-space is necessary in the preceding proposition. Indeed, the space X in Example 1.6 is an almost locally compact, non-q-space (hence a non-locally compact space) such that $(\mathcal{P}, \tau_C)$ is a Baire space (see Theorem 4.2).

5. An application

Let (X,d) be a metric space. For $B \in CL(X)$ and $f \in C(B,\mathbb{R}^n)$ let $\Gamma(f,B)$ denote the graph of the partial function $(B,f) \in \mathcal{P}$; further, let $\mathcal{G} = \{\Gamma(f,B)\colon (B,f) \in \mathcal{P}\}$. For compact $K \subset X$ and $\Gamma(f,B), \Gamma(g,C) \in \mathcal{G}$ define

$$\begin{aligned}&\rho_K\big(\Gamma(f,B),\Gamma(g,C)\big)\\ &\quad = \max\big\{e(\Gamma(f,B\cap K),\Gamma(g,C)), e(\Gamma(g,C\cap K),\Gamma(f,D))\big\},\end{aligned}$$

where e is the excess functional on $X \times \mathbb{R}^n$ induced by the box metric of d and the Euclidean metric on $\mathbb{R}^n$. A net $\{\Gamma(f_s,B_s) \in \mathcal{G}\colon s \in \Sigma\}$ is said to be τ_G-convergent to $\Gamma(f_0,B_0) \in \mathcal{G}$ (see [6,7]), provided for each $K \in \mathcal{K}(X)$ the numerical net $\{\rho_K(\Gamma(f_0,B_0),\Gamma(f_s,B_s))\colon s \in \Sigma\}$ converges to zero. Clearly, the Hausdorff metric convergence in $\mathcal{G}$ implies τ_G-convergence and the two coincide if X is compact.

It was shown in [6], that after identifying partial functions with their respective graphs, τ_G-convergence is always topological; in particular, the generalized compact-open topology τ_C topologizes τ_G if X is locally compact. Therefore, in view of our Corollary 4.4(i) and Proposition 1.8 we have

Theorem 5.1. *Let X be a locally compact metric space. Then $(\mathcal{G},\tau_G)$ is weakly α-favorable and hence a Baire space.*

Remark 5.2. Note that, if X is a hemicompact metrizable space, then $(\mathcal{G},\tau_G)$ is a Polish space (cf. [19, Theorem 2.8]).

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